Big-O, Omega, Theta Notes

1. Definition:

Big-O: $f(n) \in O(g(n))$ iff $\exists c > 0$,
$\exists n_0 > 0 \in \mathbb{N}$ s.t. $\forall n \geq n_0$,
$f(n) \leq c \cdot g(n)$.

Big-Omega (Ω): $f(n) \in \Omega(g(n))$
iff $\exists c > 0$,
$\exists n_0 > 0 \in \mathbb{N}$ s.t.
$\forall n \geq n_0$, $f(n) \geq c \cdot g(n)$

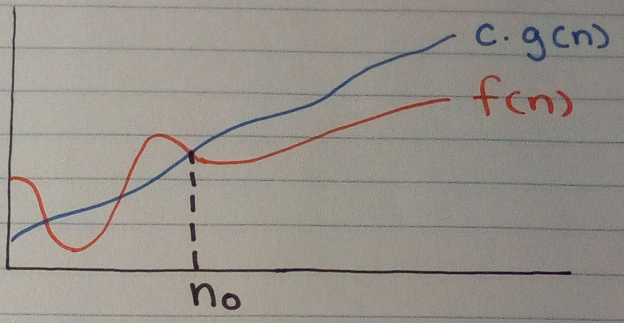
Big-Theta (Θ): $f(n) \in \Theta(g(n))$ iff
$f(n) \in O(g(n))$ and
$f(n) \in \Omega(g(n))$.

Note: $f(n) \in \Theta(g(n))$ iff $f(n) \in O(g(n))$
and $g(n) \in O(f(n))$.

Note: $f(n) \in O(g(n)) \Longleftrightarrow g(n) \in \Omega(f(n))$

Note: If $f(n) \in O(g(n))$ and $g(n) \in$
$O(h(n))$, then $f(n) \in O(h(n))$.

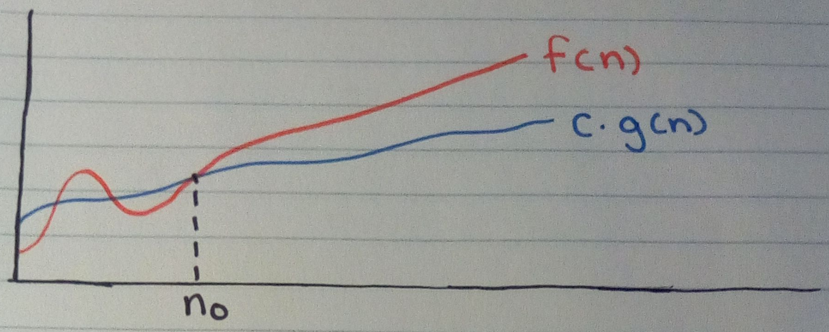
Big-O:



This shows $f(n) \in O(g(n))$. Big-O provides an upper bound. If an algorithm is $O(x)$, then the alg takes at most $c \cdot x$ steps to run on the worst possible input.

To show if an alg is $O(x)$, we have to show that for every input, the alg takes at most $c \cdot x$ steps. We can do this by overestimating the steps it takes.

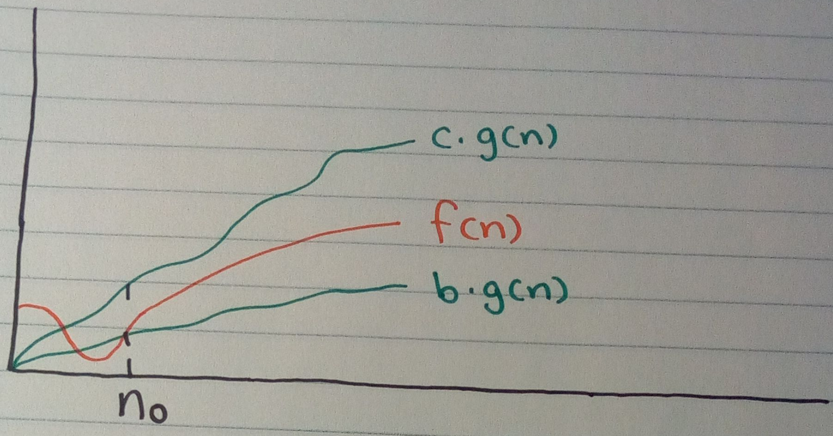
Big-Omega (Ω)



This shows $f(n) \in \Omega(g(n))$.
Big-Omega provides a lower bound. If an alg is $\Omega(x)$, then the alg takes at least $c \cdot x$ steps to run on the worst possible input.

To show if an alg is $\Omega(x)$, we have to show that there is an input that makes the alg take at least $c \cdot x$ steps.

Big-Theta (Θ):



The picture shows $f(n) \in \Theta(g(n))$.
Big-Theta provides a tight bound.

2. Examples of Big-O, Omega, Theta Using Definition:

a) Show $12n^2 + 10n + 10 \in \Theta(n^2)$

Soln:

We have to show both:

1. $12n^2 + 10n + 10 \in O(n^2)$
2. $12n^2 + 10n + 10 \in \Omega(n^2)$

To show the first, we do

$$12n^2 + 10n + 10 \leq 12n^2 + 10n^2 + 10$$
$$= 23n^2 + 10$$
$$\leq 24n^2, \quad n \geq \sqrt{10}$$

$\therefore 12n^2 + 10n + 10 \in O(n^2)$

To show the second, we do:

$$12n^2 + 10n + 10 > 12n^2 + 10n$$
$$\geq 12n^2, \quad n \geq 0$$

$\therefore 12n^2 + 10n + 10 \in \Omega(n^2)$

$\therefore 12n^2 + 10n + 10 \in \Theta(n^2)$

b) Show $n^3 - n^2 + 5 \in \Theta(n^3)$

Soln:

$$n^3 - n^2 + 5 \leq n^3 + 5$$
$$\leq n^3 + 5n^3, \quad n \geq 1$$
$$= 6n^3$$

$\therefore n^3 - n^2 + 5 \in O(n^3)$

$$n^3 - n^2 + 5 > n^3 - n^2$$
$$\geq bn^3$$

Dividing both sides by n^2:

$$n - 1 \geq bn$$
$$n - bn \geq 1$$
$$n(1-b) \geq 1$$
$$n \geq \frac{1}{1-b}$$

Since $n \geq n_0$ and $n_0 \in \mathbb{N}$, $b \leq 1$.

Let $b = \frac{1}{2}$, $n = 2$.

$\therefore n^3 - n^2 + 5 \in \Omega(n^3)$

$\therefore n^3 - n^2 + 5 \in \Theta(n^3)$

3. Chart of Functions

In the chart on the next page, if you see a y, it means that the function in the row is in Big-o of the function in the column.

Note: $\lg(n)$ means $\log_2 n$ and $\ln(n)$ means $\log_e n$.

Note: $\ln(n) \in O(\lg(n))$ and $\lg(n) \in O(\ln(n))$ because the base doesn't matter. We can prove this using the change of base theorem.

$$\log_2 n = \frac{\log_e n}{\log_e 2} \rightarrow \lg(n) \in O(\ln(n))$$

$$\log_e n = \frac{\log_2 n}{\log_2 e} \rightarrow \ln(n) \in O(\lg(n))$$

Note: $\lg(n^2) = 2\lg(n)$

Note: $2^{3n} = 8n$

	$\ln(n)$	$\lg(n)$	$\lg(n^2)$	$(\lg(n))^2$	n	$n\lg n$	2^n	2^{3n}
$\ln(n)$	Y	Y	Y	Y	Y	Y	Y	Y
$\lg(n)$	Y	Y	Y	Y	Y	Y	Y	Y
$\lg(n^2)$	Y	Y	Y	Y	Y	Y	Y	Y
$(\lg(n))^2$				Y	Y	Y	Y	Y
n					Y	Y	Y	Y
$n\lg(n)$						Y	Y	Y
2^n							Y	Y
2^{3n}								Y

4. Using Limits to Prove or Disprove Big-O.

Thm: If $\lim_{n\to\infty} \frac{f(n)}{g(n)} \in [0, \infty)$, then $f(n) \in O(g(n))$.

To prove that $f(n) \in O(g(n))$, we have to show that $\lim_{n\to\infty} \frac{f(n)}{g(n)}$ exists and is finite.

To disprove that $f(n) \in O(g(n))$, we have to show that $\lim_{n\to\infty} \frac{f(n)}{g(n)} = \infty$.

Thm: If $\lim_{n\to\infty} \frac{f(n)}{g(n)} \in (0, \infty]$, then $f(n) \in \Omega(g(n))$.

Thm: If $\lim_{n\to\infty} \frac{f(n)}{g(n)} \in (0, \infty)$, then $f(n) \in \Theta(g(n))$.

Note: If $\lim_{n\to\infty} \frac{f(n)}{g(n)}$ DNE and is not ∞, then there is no conclusion. This happens with piecewise functions.

5. Examples of Using Limits to Prove or Disprove Big-O:

a) Prove $\frac{n(n+1)}{2} \in O(n^2)$

Soln:

$$\lim_{n \to \infty} \frac{\frac{n(n+1)}{2}}{n^2}$$

$$= \lim_{n \to \infty} \frac{n(n+1)}{2n^2}$$

$$= \lim_{n \to \infty} \frac{n^2+n}{2n^2}$$

$$= \frac{1}{2}$$

$$\therefore \frac{n(n+1)}{2} \in O(n^2)$$

b) Prove $\ln(n) \in o(n)$

Soln:

$$\lim_{n \to \infty} \frac{\ln(n)}{n}$$

$$= \lim_{n \to \infty} \frac{\frac{1}{n}}{1} \quad \text{L'hopital}$$

$$= \lim_{n \to \infty} \frac{1}{n}$$

$$= 0$$

$\therefore \ln(n) \in o(n)$

c) Prove $(\lg(n))^2 \in o(8^n)$

Soln:

$$\lim_{n \to \infty} \frac{(\lg(n))^2}{8^n}$$

$$= \lim_{n \to \infty} \frac{2(\lg(n))}{(\ln 8)(n)(8^n)} \quad \text{L'hopital}$$

$$= \frac{2}{\ln 8} \lim_{n \to \infty} \frac{\lg(n)}{n(8^n)}$$

$$= \frac{2}{\ln 8} \lim_{n \to \infty} \frac{1}{(n^2)(8^n)(\ln 8)} \quad \text{L'hopital}$$

$$= \frac{2}{(\ln 8)^2} \lim_{n \to \infty} \frac{1}{(n^2)(8^n)}$$

$$= 0$$

$\therefore (\lg(n))^2 \in o(8^n)$

d) Disprove $n \in o(\ln(n))$

Soln:

$$\lim_{n \to \infty} \frac{n}{\ln(n)}$$

$$= \lim_{n \to \infty} \frac{1}{1/n} \quad \text{L'hopital}$$

$$= \lim_{n \to \infty} n$$

$$= \infty$$

$\therefore n \notin o(\ln(n))$

6. How can Big-o be abused?

Consider the 2 statements below:

1. $10^{100} n \in O(n)$
2. $n + 10^{100} \in O(n)$

These are not practical alg times, but O, Θ can't detect them. This is a price for ignoring machine differences. These cases are rare. O and Θ are usually informative.